Photonic Crystals:
Periodic Surprises in Electromagnetism

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To Begin: A Cartoon in 2d

\[ \vec{E}, \vec{H} \sim e^{i(\vec{k} \cdot \vec{x} - \omega t)} \]

\[ |\vec{k}| = \frac{\Delta x}{c} = \frac{2\Delta x}{\Pi} \]
To Begin: A Cartoon in 2d

planewave

\[ \vec{E}, \vec{H} \sim e^{i(\vec{k} \cdot \vec{x} - \omega t)} \]

\[ |\vec{k}| = \frac{\lambda}{c} = \frac{2\lambda}{\Pi} \]

for most \( \Pi \), beam(s) propagate through crystal without scattering (scattering cancels coherently)

...but for some \( \Pi (\sim 2a) \), no light can propagate: a photonic band gap
Photonic Crystals

periodic electromagnetic media

1887
1-D periodic in one direction

1987
2-D periodic in two directions
3-D periodic in three directions

(need a more complex topology)

with photonic band gaps: “optical insulators”
Photonic Crystals

periodic electromagnetic media

can trap light in cavities and waveguides (“wires”)

magical oven mitts for holding and controlling light

with photonic band gaps: “optical insulators”
Photonic Crystals

periodic electromagnetic media

But how can we understand such complex systems?
Add up the infinite sum of scattering? Ugh!
A mystery from the 19th century

\[ \vec{E} \rightarrow \]

conductive material

\[ \vec{J} = \nabla \vec{E} \]

current: conductivity (measured)

mean free path (distance) of electrons
A mystery from the 19th century

crystalline conductor (e.g. copper)

$e^- \rightarrow \vec{E}$

10’s of periods!

current: $\vec{J} = \int \vec{E}$

conductivity (measured)

mean free path (distance) of electrons
A mystery solved…

1. electrons are waves (quantum mechanics)

2. waves in a periodic medium can propagate without scattering:

   Bloch’s Theorem (1d: Floquet’s)

The foundations do not depend on the specific wave equation.
Time to Analyze the Cartoon

planewave

\( \vec{E}, \vec{H} \sim e^{i(\vec{k} \cdot \vec{x} - \omega t)} \)

\[ |\vec{k}| = \frac{\Box}{c} = \frac{2\Box}{\Pi} \]

...but for some \( \Box (\sim 2a) \), no light can propagate: a photonic band gap

for most \( \Box \), beam(s) propagate through crystal without scattering (scattering cancels coherently)
Fun with Math

\[ \vec{E} = \frac{1}{c} \frac{\partial}{\partial t} \vec{H} = i \frac{\partial}{\partial c} \vec{H} \]

First task: get rid of this mess

\[ \vec{H} = \frac{1}{c} \frac{\partial}{\partial t} \vec{E} + J^0 = i \frac{\partial}{\partial c} \vec{E} \]

dielectric function \( \square(x) = n^2(x) \)

\[ \frac{1}{\square} \frac{\partial}{\partial \square} \vec{H} = \frac{\partial}{\partial c} \vec{H} + \text{constraint} \]

- **eigen-operator**
- **eigen-value**
- **eigen-state**
Hermitian Eigenproblems

\[ \hat{H} \hat{c} = \hat{c} \hat{H} = w \hat{c} \]

+ constraint

\[ \hat{c} \cdot \hat{H} = 0 \]

Hermitian for real (lossless)

well-known properties from linear algebra:

\[ w \text{ are real (lossless)} \]

\[ \hat{c} \text{ are orthogonal} \]

\[ \hat{c} \text{ are complete (give all solutions)} \]
Periodic Hermitian Eigenproblems


if eigen-operator is periodic, then Bloch-Floquet theorem applies:

\[
\hat{H}(\vec{x}, t) = e^{i(\vec{k} \cdot \vec{x} - \omega t)} \hat{H}_\vec{k}(\vec{x})
\]

Corollary 1: \( \vec{k} \) is conserved, i.e. no scattering of Bloch wave

Corollary 2: \( \hat{H}_\vec{k} \) given by finite unit cell, so \( \square \) are discrete \( \square_n(\vec{k}) \)
Periodic Hermitian Eigenproblems

Corollary 2: $\tilde{H}_k$ given by finite unit cell, so the states $\square$ are discrete $\square_n(k)$

band diagram (dispersion relation)

map of what states exist & can interact

range of $k$?
**Periodic Hermitian Eigenproblems in 1d**

\[ H(x) = e^{ikx} H_k(x) \]

Consider \( k + 2\pi/a \):

\[ e^{i(k+2\pi/a)x/a} H_{k+2\pi/a}(x) = e^{ikx} e^{i2\pi x/a} H_{k+2\pi/a}(x) \]

\( k \) is periodic:

\( k + 2\pi/a \) equivalent to \( k \)

“quasi-phase-matching”

periodic!

satisfies same equation as \( H_k \)

\[ = H_k \]
**Periodic Hermitian Eigenproblems in 1d**

$k$ is periodic:

$k + 2\pi/a$ equivalent to $k$

“quasi-phase-matching”

\[
\begin{align*}
\psi(x) &= \psi(x+a) \\
E(x) &= E(x+a)
\end{align*}
\]

** irreducible Brillouin zone**
Any 1d Periodic System has a Gap


Start with a uniform (1d) medium:

\[ \square = \frac{k}{\sqrt{\square}} \]
Any 1d Periodic System has a Gap

[Treat it as “artificially” periodic]

bands are “folded” by $2\pi/a$ equivalence

\[ e^{i(x + a)} = e^{i x}. \]
Any 1d Periodic System has a Gap


Treat it as

“artificially” periodic
Any 1d Periodic System has a Gap


Add a small “real” periodicity

\[ b_2 = b_1 + \sin \pi a x \]

\[ \cos \pi a x \]

\[ b(x) = b(x+a) \]

\[ x = 0 \]
Any 1d Periodic System has a Gap


Add a small “real” periodicity

\[ \ell_2 = \ell_1 + \ell_1 \]

Splitting of degeneracy:
state concentrated in higher index (\( \ell_2 \)) has lower frequency

\( \ell(x) = \ell(x + a) \)
Some 2d and 3d systems have gaps

- In general, eigen-frequencies satisfy \( \text{Variational Theorem} \):

\[
\Box_1(\vec{k})^2 = \min_{\vec{E}_1} \frac{\left( \Box + i\vec{k} \right) \cdot \vec{E}_1}{c^2} \quad \text{“kinetic”}
\]

\[
\Box_1(\vec{k})^2 = \min_{\vec{E}_1} \frac{\left( \Box + i\vec{k} \right) \cdot \vec{E}_1}{c^2} \quad \Box \cdot \vec{E}_1 = 0
\]

\( \Box \cdot \vec{E}_1 = 0 \)

\[ \Box_1(\vec{k})^2 = \min_{\vec{E}_1} \quad \text{inverse “potential”} \]

\[
\Box_2(\vec{k})^2 = \min_{\vec{E}_2} "\cdots" \text{ bands “want” to be in high-} \Box
\]

\[
\Box_2(\vec{k})^2 = \min_{\vec{E}_2} "\cdots" \text{ bands “want” to be in high-} \Box
\]

\( \Box \cdot \vec{E}_2 = 0 \)

\( \Box \vec{E}_1^* \cdot E_2 = 0 \) … but are forced out by \( \text{orthogonality} \)

\( \implies \text{band gap (maybe)} \)
algebraic interlude

algebraic interlude completed…

… I hope you were taking notes*

[ *if not, see e.g.: Joannopoulos, Meade, and Winn, Photonic Crystals: Molding the Flow of Light ]
2d periodicity, $\square = 12:1$

irreducible Brillouin zone

photonic band gap for $n > \sim 1.75:1$
2d periodicity, $\mathbb{C}=12:1$

$E_z$

(+ 90° rotated version)

$E_z$

$E$ $H$

Photonic Band Gap

TM bands

gap for $n > \sim 1.75:1$
2d periodicity, $a = 12:1$

irreducible Brillouin zone

$\overrightarrow{k}$
2d photonic crystal: TE gap, $\square = 12:1$

gap for $n > \sim 1.4:1$
3d photonic crystal: complete gap, $\varepsilon=12:1$

I. rod layer
II. hole layer

I: rod layer
II: hole layer

gap for $n > \sim 4:1$

You, too, can compute photonic eigenmodes!

MIT Photonic-Bands (MPB) package:
http://ab-initio.mit.edu/mpb

on Athena:
add mpb